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On 2-regular subgraphs in polyhedral graphs

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Abstract

We show that every polyhedral graph G contains a 2-regular subgraph U such that $G-U$ is a forest of trees with at most three leaves. With this tool we give a partial solution of a problem posed in a problem session at the Workshop "Cycles and Colourings '98" in Stará Lesná, Slovakia. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

For a non-negative integer k a subgraph U of a graph G is called *k-dominating* if for any vertex $v\in V(G)$ there is a vertex $u\in V(U)$ with $d_G(v,u)\leqslant k$, where $d_G(v,u)$ is the usual distance between v and u in G. Note, that a 2-factor is a 0-dominating 2-regular subgraph. In [5], Harant posed the problem for which graphs there is a 1-dominating 2-regular subgraph. More generally, this paper summarizes our knowledge about k-dominating 2-regular subgraphs. The main result of the paper is

Theorem 1.1. *Every 3-connected planar graph G contains a 2-regular subgraph U such that every component of $G-U$ is a tree with at most three leaves.*

A subgraph U of a 3-connected planar graph G as in Theorem 1.1 is 2-dominating (but not necessarily 1-dominating, see Example 1). Furthermore, if the minimum degree of G is >3, U is a 1-dominating subgraph of G (but not necessarily a 2-factor, see Example 2). The proof of Theorem 1.1 is based on a generalization of Tutte's Theorem [8] by Thomas and Yu [6] which is stated in Section 2.

For an arbitrary integer n there are 2-connected, but not 3-connected planar graphs, as well as k-connected (not necessarily planar) graphs with arbitrarily large k, without an n-dominating 2-regular subgraph (see Examples 3 and 4).

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On the other hand, by Tutte's Theorem 4-connected planar graphs are Hamiltonian and by a generalization of Thomassen [7] even every essentially 4-connected planar graph (a graph G is essentially 4-connected if it is 3-connected, but every 3-separator is the neighborhood of a vertex of G) contains a 1-dominating cycle (but not neccesarily a 2-factor, see Example 5). As a consequence of Petersen's Theorem that every bridgeless cubic graph has a 1-factor, [3], every such graph has a 2-factor, too. Finally, we mention the following result which is easy to prove.

Theorem 1.2. *Every graph G with the property, that every $v \in V(G)$ has two adjacent neighbors, contains a* 1*-dominating subgraph U consisting of disjoint* 3*-cycles.*

Theorem 1.2 holds because every maximal subgraph U, consisting of disjoint 3-cycles, of G is a 1-dominating subgraph of G. Otherwise, there would be a vertex $v \in V(G)$ such that $d_G(v, U) \geqslant 2$ and there would be a 3-cycle on v and 2 adjacent neighbors of v in $G - U$. A consequence of Theorem 1.2 is that triangulations of any surface have a 1-dominating system of disjoint triangles (but not necessarily a 2-factor, see Examples 2 and 5).

In Section 2, we will prove Theorem 1.1 and in Section 3, we will give the examples mentioned above.

2. Notation and proof of Theorem 1.1

Throughout this paper, we consider finite plane graphs with no loops or multiple edges, and use the notation and terminology of [3], unless otherwise stated. For an arbitrary integer k a *k-separator* of a graph G is a separating set S with $|S| = k$. For a plane graph G let $A(G)$ be the subgraph induced by the boundary of the outer face of G and an *endblock* be a leaf of the block graph of G. For a subgraph H of G a vertex $x \in V(G - H)$ is called a *G-neighbor* of H if x has at least one neighbor in $V(H)$. Let $\mathscr{P}$ be the class of all 3-connected plane graphs and $\mathscr{P}_1$ be the class of all graphs obtained from graphs of $\mathscr{P}$ by deleting a vertex incident with their outer face. Note, that $\mathscr{P}_1$ equals the set of *circuit graphs* as defined in [4].

Furthermore, for an arbitrary integer $n > 1$ let $\mathscr{P}_n$ be the set of plane graphs G such that

(i) all blocks of G are isomorphic to K^2 or belong to $\mathscr{P}_1$,
(ii) the graph has at most n endblocks,
(iii) for every block B of G we have $A(B) \subseteq A(G)$.

Let $G \in \mathscr{P}_1$, and let $D \subseteq G$ be a cycle of G. We call D an *OTC* (for Outer–Tutte-Cycle, see also [8]) if for every component C of $G - D$ with $V(C \cap A(G)) = \emptyset$ the number of their neighbors in G is 3, and for all other components of $G - D$ the number of its neighbors in G is 2.

In the next step we will discuss some properties of $\mathscr{P}_n$.

Lemma 2.1. *For $G_1 \in \mathscr{P}_1$ let G be a 2-connected subgraph of G_1 such that $G_1 - G$ is embedded in the outer face of G. Then $G \in \mathscr{P}_1$.*

Proof. Let G_0 be a graph of $\mathscr{P}$ such that G_1 is obtained from G_0 by deleting the vertex x_1 of $V(A(G_0))$. Thus, x_1 belongs to the outer face of G. Let G' be obtained from G by connecting x_1 with all vertices of $A(G)$. Suppose for a contradiction that S is a 2-separator of G'. Then $x_1 \notin S$ because G is 2-connected. Thus, $G - S$ would have a component disjoint from $A(G)$. All G_0-neighbors of this component are contained in S. Consequently, S is a 2-separator of G_0, a contradiction. □

Lemma 2.2. *Let $G_0 \in \mathscr{P}$, $G \subset G_0$ such that $G_0 - G$ is embedded in the outer face of G, S be the set of G_0-neighbors of G, and X be a cycle such that $G_0 \cup X$ is plane and $S \subseteq V(X)$. Then $G \in \mathscr{P}_{|S|}$.*

Proof. If $|S| = 1$, by the definition of $\mathscr{P}_1$ Lemma 2.2 holds. Thus, we assume $|S| > 1$. By Lemma 2.1 all blocks of G, not isomorphic to K^2, belong to $\mathscr{P}_1$. Because G_0 is 3-connected, every endblock of G has at least two G_0-neighbors in $G_0 - G$. Let G' be the following graph. The vertex set of G' contains the endblocks of G, the vertices in S and two new vertices a and b. We connect an endblock B of G with a vertex $x \in S$ if x is a G_0-neighbor of B. To complete the edge set of G', we connect all vertices in S with a and all endblocks of G with b. Because $G_0 \cup X$ is plane, G' is planar. Furthermore, G' is bipartite. Let l be the number of endblocks of G, then $|V(G')| = |S| + l + 2$. Because every endblock of G has two G_0-neighbors, $|E(G')| \geqslant l + 2l + |S| = 3l + |S|$.

Because G' is bipartite, planar and simple, $|E(G')| \leqslant 2|V(G')| - 4$.

Thus, $3l + |S| \leqslant |E(G')| \leqslant 2|V(G')| - 4 = 2(|S| + l + 2) - 4$. Consequently, $l \leqslant |S|$. □

A simple corollary of Lemma 2.2 is that every graph G_2, obtained from a graph $G_1 \in \mathscr{P}_1$ by deleting any vertex of $A(G_1)$, is contained in $\mathscr{P}_2$.

Lemma 2.3. *For an arbitrary integer $k > 2$* let *$G_k \in \mathscr{P}_k$ and $x_{k+1} \in V(A(G_k))$ be a vertex of an endblock $B \in \mathscr{P}_1$ of G_k, but not a cutvertex of G_k. Then $G_{k+1} := G_k - x_{k+1} \in \mathscr{P}_{k+1}$.*

Proof. By Lemma 2.1 all blocks of G_{k+1}, not isomorphic to K^2, belong to $\mathscr{P}_1$. We get the block graph of G_k from the block graph of G_{k+1} by substitution of the block graph of $B - x_{k+1}$ by one vertex, namely B. By Lemma 2.2, $B - x_{k+1} \in \mathscr{P}_2$. Thus, the block graph of $B - x_{k+1}$ has at most two endblocks. The block graph of G_k has at most $k - 1$ endblocks different from B. Thus, the block graph of G_{k+1} has at most $k + 1$ endblocks. □

A simple corollary of Theorem (2.7) in [6] is the following one:

Corollary 2.4. *Let $G \in \mathscr{P}_1$, and $F \subseteq V(A(G))$ with $|F| \leqslant 3$. Then there is an OTC D in G with $F \subseteq V(D)$.*

For the graphs of $\mathscr{P}_3$ we get the following result.

Theorem 2.5. *Let $G_3 \in \mathscr{P}_3$ be not a tree. Then there is a cycle $D \subseteq G_3$ such that all components of $G_3 - D$ belong to $\mathscr{P}_3$.*

Proof. Let y be a vertex of the block graph of G_3 having maximum degree, B be a block, not isomorphic to K^2, of G_3 with minimum distance to y in the block graph of G_3, and F be the set of cutvertices of G_3 contained in $V(B)$. Then $|F| \leqslant 3$ and $F \subseteq A(G)$. By Corollary 2.4 B has an OTC D with $F \subseteq V(D)$. We have to show that every component of $G_3 - D$ is contained in $\mathscr{P}_3$. If there is no such component, Theorem 2.5 holds. Let C be a component of $G_3 - D$.

Let $B_0 \in \mathscr{P}_3$ be a graph, B is obtained from, by deleting one vertex x_1. For a component $C \subseteq B$ let S be the set of B_0-neighbors of C. Because S is a 3-separator of B_0 and $B_0 \in \mathscr{P}$, there is a cycle X on the vertices of S, such that $B_0 \cup X$ is plane. By Lemma 2.2 $C \in \mathscr{P}_3$.

On the other hand, if $C \nsubseteq B$ then C is a component of $G_3 - B$. If C contains only one G_3-neighbor of B, then its block graph is a subgraph of the block graph of G_3 and thus $C \in \mathscr{P}_3$, otherwise there is a cutvertex $x_3 \in V(B)$ of G_3 and an induced subgraph $G_2 \in \mathscr{P}_2$ containing x_3 in an endblock with $C = G_2 - x_3$. By Lemma 2.3 we have $C = G_2 - x_3 \in \mathscr{P}_3$. □

Proof of Theorem 1.1. Theorem 2.5 implies the following greedy algorithm. Let $G \in \mathscr{P}_3$, $H = G$ and U be the empty graph. As long as a component of H contains a cycle D as in Theorem 2.5 set $H = H - D$ and $U = U \cup D$. The algorithm results in some graphs U and H with the following properties. $H = G - U$, U is 2-regular and by Theorem 2.5 every component of H is a tree with at most three leaves. Since every 3-connected plane graph belongs to $\mathscr{P}_3$, this completes the proof of Theorem 1.1. □

3. Graphs without *n*-dominating 2-regular subgraphs

To construct such graphs we use the following simple observation.

Lemma 3.1. *Let G be a graph containing a subgraph S such that $G - S$ has $m \geqslant |V(S)| + 1$ components C_i $(i = 1, 2, \ldots, m)$ and every C_i contains a vertex v_i such that for every vertex x of S or of a cycle in C_i, $d_G(v_i, x) > n$ holds. Then G has no n-dominating 2-regular subgraph.*

If such a graph G would have a n-dominating 2-regular subgraph U, $U - S$ would have at most $|V(S)|$ components being paths. On the other hand, every component

C_i $(i=1,2,\ldots,m)$ of $G-S$ contains a component of $U-S$ being a path—a contradiction.

Example 1 (*Böhme and Harant* [1]). Let S be a plane triangulation with more faces than $|V(S)|$. Insert a 'green' vertex into each triangle. Double each edge and subdivide each of the new edges by a 'red' vertex. Now connect each of the red vertices with the green vertex in the incident hexagon. The result is a 3-connected plane graph G without a 1-dominating 2-regular subgraph .

Example 2. Suppose that the vertices of S in Example 1 were green and let G' be the following graph obtained from the graph G of Example 1. Connect every two green vertices incident with a common face of G. Now insert in every triangle T with two vertices in $V(S)$ another plane copy I of the icosahedron such that $T=A(I)$. G' is a plane triangulation with minimum degree 5, having no 2-factor.

Example 3 (*Bonsma and Kratochvil* [2], *Böhme and Harant* [1], *independently*). Let S be the complete graph on k vertices and for $i=1,2,\ldots,k+1$ let C_i be a tree with degrees 1 and k, containing a vertex v such that $d_{C_i}(v,x)\geqslant n$ for all leaves x. Let G be the graph obtained from $S\cup\bigcup_{i=1}^{k+1} C_i$ by connecting all leaves of all C_i with all vertices of S. G is k-connected and has no n-dominating 2-regular subgraph.

Example 4. Let T be a tree with leaf set L, vertex degree 5 for every vertex of $T-L$, containing a vertex v such that $d_T(v,x)=2n+2$ for every $x\in L$. Let H be a connected graph with exactly one cutvertex w such that $V(H\cup T)=L$, every block of H is isomorphic to the icosahedron graph and contains exactly one leaf of T, and $d_H(w,x)=1$ for every $x\in L$. The graph $G=H\cup T$ is 2-connected, planar, has minimum degree 5, but no n-dominating 2-regular subgraph.

Example 5. In any surface there is an embedded graph S with more faces than $|V(S)|$ such that every face of S is homeomorphic to a plane disc. We get a triangulation G without 2-factor of this surface by adding one new vertex v in every face and connecting v with the vertices incident with this face. If S is 4-connected, then G is essentially 4-connected.

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